ODE Refresher

1st-order linear

$$y'(t) + p(t)\,y(t) = q(t)$$

integrating factor

$$\mu(t) = e^{\int p(t)\,dt}$$

leading coefficient is 1

multiply the ODE by $\mu(t)$

$$\underbrace{\mu y' + p\mu y}_{\frac{d}{dt}(\mu y)} = q\mu$$

$$\frac{d}{dt}(\mu y) = \mu q$$

integrate both sides and solve for y

example $\quad y' + \frac{2}{t}y = \frac{\cos(t)}{t^2} \quad (t > 0)$

$$\mu = e^{\int \frac{2}{t}\,dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

$$t^2\left(y' + \frac{2}{t}y\right) = t^2 \cdot \frac{\cos(t)}{t^2}$$

$$\underbrace{t^2 y' + 2ty}_{} = \cos(t)$$

$$\frac{d}{dt}\left(t^2 y\right) = \cos(t)$$

integrate both sides

$$t^2 y = \sin(t) + C$$

$$\boxed{y = \frac{\sin(t)}{t^2} + \frac{C}{t^2}}$$

it will show up later when we solve the heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \qquad u = u(x,t)$$

there is a 1st-order linear that needs to be solved

$$T'(t) + \alpha^2 \lambda T(t) = 0 \qquad \alpha, \lambda \text{ constants} > 0$$

this can also be solved as a separable eq.

$$\frac{dT}{dt} = -\alpha^2 \lambda T$$

$$\frac{1}{T} dT = -\alpha^2 \lambda \, dt$$

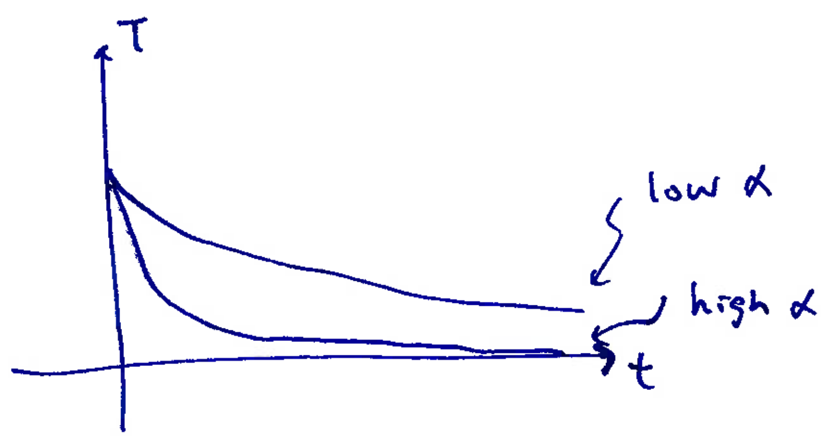
$$\ln|T| = -\alpha^2 \lambda t + c$$

constant

$$T(t) = e^{-\alpha^2 \lambda t + c} = e^{-\alpha^2 \lambda t} e^{c}$$

$$\boxed{T(t) = C e^{-\alpha^2 \lambda t}}$$

in the context of heat eq, this tells us how heat dissipates as a function of time for a given λ and position.



2nd-order homogeneous

$$y'' + ky = 0$$

Solutions are of the form e^{rt}

r is solution of the characteristic eq.

$$r^2 + k = 0$$

$$r^2 = -k$$

if $k = 0$, $r = 0, 0$, $y = C_1 + C_2 t$ (line)

if $k > 0$, $r = \pm\sqrt{k} = \pm i\sqrt{|k|}$ $\quad y = \cos(\sqrt{k}\,t)$, $y = \sin(\sqrt{k}\,t)$

$$y = C_1 \cos(\sqrt{k}\,t) + C_2 \sin(\sqrt{k}\,t) \quad \text{(bounded)}$$

if $k < 0$, $r = \pm\sqrt{|k|}$ $\quad y = e^{\sqrt{|k|}t}$, $y = e^{-\sqrt{|k|}t}$

$$y = C_1 e^{\sqrt{|k|}t} + C_2 e^{-\sqrt{|k|}t} \quad \text{(unbounded)}$$

$$y = d_1 \cosh(\sqrt{k}\,t) + d_2 \sinh(\sqrt{k}\,t)$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

this shows up when we solve the heat, wave, and Laplace's eqs.

uppercase X

$$X''(x) + \lambda X(x) = 0 \qquad \text{or} \qquad Y''(y) + \lambda Y(y) = 0$$

solutions:

$$X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x) \qquad \text{if } \lambda > 0$$

$$X(x) = c_1 \cosh(\sqrt{\lambda}\, x) + c_2 \sinh(\sqrt{\lambda}\, x) \qquad \text{if } \lambda < 0$$

$$X(x) = c_1 + c_2 x \qquad \text{if } \lambda = 0$$

in the context of heat eq, this governs the space part of the solution (how temperature varies w/ position if time is fixed)

now briefly about nonhomogeneous 2nd-order in the context of mass-spring system: $mx'' + kx = f(t)$

when $f(t) = F_0 \sin(\omega t)$ (sinusoidal input)

ω : input frequency

F_0 : input magnitude

w/o input: $mx'' + kx = 0$

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$ (complementary solution)

w/ input: $$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right) + x_p$$

particular solution

$\sqrt{\frac{k}{m}} = \omega_0$ (natural frequency)

if $\omega \neq \omega_0$ (input freq $\neq$ natural freq) we can solve using method of undetermined coefficients

$$m x'' + k x = F_0 \sin(\omega t)$$

$$x_p = A\cos(\omega t) + B\sin(\omega t)$$

sub into the ODE

$\vdots$

$$A = 0, \quad B = \frac{F_0}{k - m\omega^2} \qquad \text{trouble if} \quad k = m\omega^2 \quad \text{or} \quad \omega = \sqrt{\frac{k}{m}}$$

$$x(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right) + \frac{F_0}{k - m\omega^2} \sin(\omega t)$$

but if input freq = natural freq $\left(\omega = \sqrt{\frac{k}{m}},\ k - m\omega^2 = 0\right)$

x_p needs to be adjusted due to duplication of complementary

$$x_p = A\underline{\underline{t}}\cos(\omega t) + B\underline{\underline{t}}\sin(\omega t)$$

sub into $m x'' + k x = F_0 \sin(\omega t)$

$\vdots$

$$B = 0, \quad A = -\frac{F_0}{2mW}$$

$$x(t) = c_1 \cos\left(\sqrt{\frac{K}{m}}\, t\right) + c_2 \sin\left(\sqrt{\frac{K}{m}}\, t\right) - \frac{F_0}{2mW}\, \underline{\underline{t}} \cos(wt)$$

goes to ∞ as t grows

<u>resonance</u>